Week 9 - Monday

COMP 2230

Last time

- Exam 2 post mortem

Questions?

Assignment 4

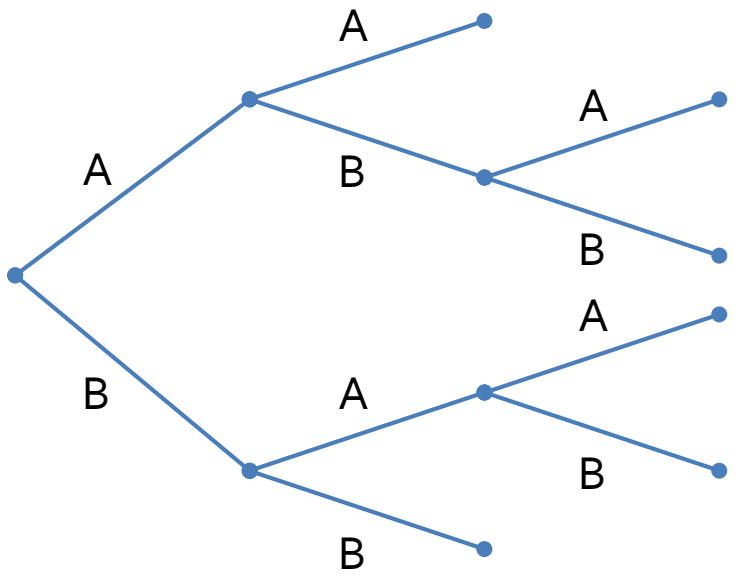
Logical warmup

- You need to pack for a midnight flight to Iceland but the power is out.
- In your closet are six pairs of shoes, six black socks, six gray socks, six pairs of brown gloves, and six pairs of tan gloves.
- Unfortunately, it's too dark to match shoes or to see any colors.
- How many of each of these items do you need to take to be sure of getting a matched pair of shoes, two socks of the same color, and gloves that match each other?

Multiplication Rule

Possibility trees

- We can use a tree to represent all the possibilities in a situation
- Example:
  - Teams A and B are playing a best of 3 tournament
  - The first team to win 2 games wins



  - How likely is it that 3 games are needed to decide the tournament, assuming that all ways of playing the tournament are equally likely (which is **not** a reasonable assumption)?

Multiplication rule

- If an operation has k steps such that
  - Step 1 can be performed in n_1 ways
  - Step 2 can be performed in n_2 ways

    …
  - Step k can be performed in n_k ways
- Then, the entire operation can be performed in $\prod_{i=1}^{k} n_i = n_1 n_2 \ldots n_k$ ways
- This rule only applies when each step always takes the same number of ways (unlike the previous possibility tree example)

Coin example

- If you flip a coin k times, how many total possibilities are there for the outcomes?

Personal Identification Numbers

- If a PIN is a 4-digit sequence, where each digit is $0-9$ or $A-Z$, how many PINs are possible?
- How many PINs are possible if no digits are repeated?
- Assuming that all PINs are equally likely, what's the probability that a PIN chosen at random has no repetitions?

Permutations

- A permutation of a set of objects is an ordering of the objects in a row
- Consider set $\{a, b, c\}$
- Its permutations are:
  - abc
  - acb
  - cba
  - bac
  - bca
  - cab
- If a set has $n \geq 1$ elements, it has $n!$ permutations

Permutations of letters in a word

- How many different ways can the letters in the word "WOMBAT" be permuted?
- How many different ways can "WOMBAT" be permuted such that "BA" remains together?
- What is the probability that, given a random permutation of "WOMBAT", the "BA" is together?
- How many different ways can the letters in "MISSISSIPPI" be permuted?
- How many would it be if we don't distinguish between copies of letters?

Permuting around a circle

- What if you want to seat 6 people around a circular table?
- If you only care about who sits next to whom (rather than who is actually in Seat 1, Seat 2, etc.) how many circular permutations are there?
- What about for n people?

Permutations of selected elements

- An r**-permutation** of a set of n element is an ordered selection of r elements from the set
- Example: A 2-permutation of $\{a, b, c\}$ includes:
  - ab
  - ac
  - ba
  - bc
  - ca
  - cb
- The number of r-permutations of a set of n elements is

$$P(n, r) = \frac{n!}{(n - r)!}$$

r-permutation examples

- What is $P(5,2)$?
- How many 4-permutations are there in a set of 7 objects?
- How many different ways can three of the letters in "BYTES" be written in a row?

Disjoint Sets

Addition rule

- If a finite set A equals the union of k distinct mutually disjoint subsets A_1, A_2, … A_k, then:

$$N(A) = N(A_1) + N(A_2) + \ldots + N(A_k)$$

Addition rule example

- How many passwords are there with length 3 or smaller?
- Assume that a password is only made up of lower case letters
- Passwords with length 3 or smaller fall into 3 disjoint sets
  - Number of passwords with length 1
  - Number of passwords with length 2
  - Number of passwords with length 3
- Total passwords $= 26 + 26^2 + 26^3 = 18278$

Difference rule

- If A is a finite set and B is a subset of A, then

$$N(A - B) = N(A) - N(B)$$

- Example:
  - Recall that a PIN has 4 digits, each of which is one of the 26 letters or one of the 10 digits
  - How many PINs contain repeated symbols?
  - What is the probability that a PIN contains a repeated symbol?

Inclusion/exclusion rule

- If A, B, C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

- And

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$

Inclusion exclusion example

- How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?
- How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

Inclusion exclusion example

- Consider a survey of 50 students about the programming languages they know
- The results are:
  - 30 know Java
  - 18 know C++
  - 26 know ML
  - 9 known both Java and C++
  - 16 know both Java and ML
  - 8 know both C++ and ML
  - 47 know at least one of the three
- How many students know none of the three?
- How many students know all three?
- How many students know Java and C++ but not ML?
- How many students know Java but neither C++ nor ML?

Pigeonhole Principle

Pigeonhole principle

- If n pigeons fly into m pigeonholes, where $n > m$, then there is at least one pigeonhole with two or more pigeons in it
- More formally, if a function has a larger domain than co-domain, it cannot be one-to-one
- We cannot say exactly how many pigeons are in any given holes
- Some holes may be empty
- But at least one hole will have at least two pigeons

Pigeonhole examples

- A sock drawer has white socks, black socks, and red argyle socks, all mixed together,
- What is the smallest number of socks you need to pull out to be guaranteed a matching pair?

- Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- If you select five distinct elements from A, must it be the case that some pair of integers from the five you selected will sum to 9?

Generalized pigeonhole principle

- If n pigeons fly into m pigeonholes, and for some positive integer k, $n > km$, then at least one pigeonhole contains $k + 1$ or more pigeons in it
- Example:
  - In a group of 85 people, at least 4 must have the same last initial

Combinations and Combinations with Repetition

Three-sentence summary

Upcoming

Next time…

- Combinations
- Combinations with repetition
- Binomial theorem and Pascal's triangle
- Expected value

Reminders

- **Keep working on Assignment 4**
- Read 9.7 and 9.8
  - Prepare a three-sentence summary
  - Extra credit if you get called on